

GAUGE THEORY AND G_2 -GEOMETRY ON CALABI-YAU LINKS

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ABSTRACT. The 7-dimensional link K of a weighted homogeneous hypersurface on the round 9-sphere in \mathbb{C}^5 has a nontrivial null Sasakian structure which is contact Calabi-Yau, in many cases. It admits a canonical co-closed G_2 -structure φ induced by the Calabi-Yau 3-orbifold basic geometry. We distinguish these pairs (K, φ) by the Crowley-Nordström \mathbb{Z}_{48} -valued ν invariant, for which we prove odd parity and provide an algorithmic formula.

We describe moreover a natural Yang-Mills theory on such spaces, with many important features of the torsion-free case, such as a Chern-Simons formalism and topological energy bounds. In fact compatible G_2 -instantons on holomorphic Sasakian bundles over K are exactly the transversely Hermitian Yang-Mills connections. As a proof of principle, we obtain G_2 -instantons over the Fermat quintic link from stable bundles over the smooth projective Fermat quintic, thus relating in a concrete example the Donaldson-Thomas theory of the quintic threefold with a conjectural G_2 -instanton count.

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1. INTRODUCTION

We propose a contemporary angle on Milnor's celebrated study of singular hypersurface links [Mil69], from the perspective of special metrics and higher-dimensional gauge theory. Our intuitive starting point was the observation that several topological properties of the Milnor fibre and its boundary the link (see Section 2) resemble those of the G_2 -invariant ν recently introduced by Crowley and Nordström [CN15], suggesting to optimists that Milnor's construction might be related to G_2 -geometry.

1.1. G_2 -metrics on Calabi-Yau links. Let $\mathcal{V} \subset \mathbb{C}^{n+1}$ be a complex analytic variety with an isolated singularity at the origin. Milnor proved that \mathcal{V} intersects transversally every sufficiently small sphere $S^{2n+1} := \partial B_\varepsilon(0)$, and the *link*

$$K := \mathcal{V} \cap S^{2n+1}$$

is a $(n-2)$ -connected smooth manifold with $\dim_{\mathbb{R}} K = 2 \dim_{\mathbb{C}} \mathcal{V} - 1$. The topologies of \mathcal{V} and of its embedding in \mathbb{C}^{n+1} are completely determined by the embedding $K \hookrightarrow S^{2n+1}$.

Suppose henceforth that $\mathcal{V} = (f)$ is an affine hypersurface defined by a homogeneous polynomial $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$, with $f(0) = 0$ and $\text{Crit}(f) \cap B_\varepsilon(0) = \{0\}$. The Hopf fibration $\pi : S^{2n+1} \rightarrow \mathbb{P}^n$ characterises the corresponding link K_f in a natural way as the total space of a S^1 -bundle over the smooth projective hypersurface V defined by f :

$$\pi : K_f \xrightarrow{S^1} V \subset \mathbb{P}^n.$$

As a circle bundle, K_f carries a global angular form $\theta \in \Omega^1(K)$, whose restriction to each fibre $\pi^{-1}(x)$ generates the cohomology $H^1(\pi^{-1}(x), \mathbb{R})$. Its exterior derivative $d\theta = -\pi^*e \in \Omega^2(K)$ is the pullback of the Euler class on the base (compare with Lemma 9).

If the link has degree $n+1$, then the projective variety V is a Calabi-Yau $(n-1)$ -fold. Fixing $n=4$, a *quintic* link K_f is a smooth Sasakian 7-manifold fibering by circles over the smooth Calabi-Yau 3-fold V , and it is the simplest example of a *Calabi-Yau (CY) link* (see Definition 12). Now, it is well-known that the Riemannian product of a Calabi-Yau 3-fold and a circle carries a torsion-free G_2 -structure, so we define naturally (see also Theorem 2.5 in [Gra69]) the following G_2 -structure on K_f :

$$(1) \quad \begin{aligned} \varphi &:= \theta \wedge \omega + \text{Im } \varepsilon, \\ \psi &:= \frac{1}{2}\omega \wedge \omega + \theta \wedge \text{Re } \varepsilon = *\varphi \end{aligned}$$

where ω and ε are respectively the Kähler and holomorphic volume forms defining the Calabi-Yau structure on V and we denote identically differential forms and their pullbacks under π . Although in the nontrivial fibration case this structure has torsion, it is still cocalibrated (see Section 2.2):

Theorem 1. *Every quintic link K_f is a 2-connected, compact, smooth real 7-manifold admitting the natural coclosed G_2 -structure (1).*

It should be stressed that Theorem 1 has recently been found and subsumed, independently, by Habib and Vezzoni [HV15, §6.2] in the context of *contact Calabi-Yau* (cCY) geometry. Their theory allows for a generalised account of this discussion for weighted homogenous links, which therefore yields many more examples of CY links, fibering over CY 3-orbifolds in weighted $\mathbb{P}^4(w)$ (see Section 2.4). This is very fortunate, because otherwise the Fermat quintic is the only strictly homogeneous quintic with an isolated singularity at the origin.

In the light of substantial recent progress in the classification of 2-connected 7-manifolds with G_2 -structures [CN14, CGN15, CN15], it is a natural task to sort such CY links (K_f, φ) . The important \mathbb{Z}_{48} -valued invariant $\nu(\varphi)$ introduced by Crowley and Nordström [CN15] allows us to distinguish such pairs, up to diffeomorphisms of K_f and homotopy of φ , but its definition is non-constructive and it requires an *ad hoc* spin coboundary 8-manifold W such that $K_f = \partial W$. In Section 3, we show that this coboundary can be

essentially taken to be a typical Milnor fibre, and we find an explicit formula for $\nu(\varphi)$ in terms of topological data:

Theorem 2. *Let $K_f \xrightarrow{S^1} V \subset \mathbb{P}^4(w)$ be a weighted Calabi-Yau link of degree d and weight $w = (w_0, \dots, w_4)$; then the Crowley-Nordström ν invariant of any S^1 -invariant G_2 -structure φ on K_f is an odd integer given by*

$$\nu(\varphi) = \left(\frac{d}{w_0} - 1\right) \dots \left(\frac{d}{w_4} - 1\right) - 3(\mu_+ - \mu_-) + 1.$$

where (μ_-, μ_+) is the signature of the intersection form on $H^4(\tilde{V}, \mathbb{R})$, for

$$\tilde{V} = \{f(z) = 1\} \subset \mathbb{C}^5.$$

Using a method by Steenbrink to calculate the signature, we obtain an effective algorithm to determine $\nu(\varphi)$ for any CY link, with straightforward computational methods (see Appendix A). We observe that several values of ν are realised in this manner, and conjecture that indeed all possible 24 values can be realised by weighted CY links. In particular, for the homogeneous case we find:

Corollary 1. *The Crowley-Nordström ν invariant of the Fermat quintic link with G_2 -structure (1) is $\nu(\varphi) = 5$.*

To the best of our knowledge, this large class of 7-manifolds with G_2 -structure of the form (K_f, φ) is the first instance besides the original reference [CN15] in which the ν invariant has been computed explicitly.

1.2. Gauge theory on contact Calabi-Yau manifolds. In Section 4 we turn to the second axis of interest in G_2 -geometry, as a model for 7-dimensional gauge theory. Since that concept appeared in the Physics literature [CDFN83], physicists pursue an analogous definition of Witten’s topological quantum field theory [Wit88] on spaces with G_2 -metrics [AOS97]. Moreover, it was noticed in [HM99] that the superpotential for M-theory compactifications on G_2 -manifolds ‘counts’ associative 3-manifolds (i.e. submanifolds calibrated by φ) in the same way as the prepotential of type II strings counts holomorphic curves in CY 3-folds. Mathematicians, on the other hand, following the seminal viewpoint of [DT98], expect the theory to culminate in a topological count of instantons, yielding an invariant for 7-manifolds with a G_2 -structure, in the same vein as the Casson invariant for flat connections over 3-manifolds [Don02]. At the current stage, however, major compactification issues remain and a more thorough analytical understanding might have to be postponed in favour of exploring a good number of examples [Cla14, SE09, Wal13, SE14, SE15, SEW15].

We propose a consistent formulation of elementary Yang-Mills theory on 7-dimensional cCY manifolds. In Section 4.1, we define a connection A on a complex vector bundle $E \rightarrow K$ to be a G_2 -instanton if $F_A \wedge \psi = 0$, where ψ is the G_2 -structure 4-form (cf. [DT98, Tia00]), which characterises A at first as a critical point of the Chern-Simons functional. In Section 4.2 we endow E with a suitable holomorphic Sasakian vector bundle structure, following the framework of Biswas and Schumacher [BS10], to obtain a notion of *Chern connection*, compatible at once with the holomorphic structure and some Hermitian bundle metric (Proposition 20). This in turn provides a secondary characteristic class leading to topological energy bounds, even though the G_2 -structure has torsion, and we prove in Section 4.3:

Theorem 3. *Let \mathcal{E} be a holomorphic Sasakian bundle over a 7-dimensional closed contact Calabi-Yau manifold M endowed with its natural cocalibrated G_2 -structure (cf. Proposition 11). If the absolute topological minimum of the Yang-Mills functional S_{YM} is attained among integrable connections, then the minima are exactly the G_2 -instantons, i.e., the critical points of the Chern-Simons functional S_{CS} .*

Finally, we show in Section 4.4 that the G_2 -instanton condition is exactly equivalent to a natural transverse Hermitian Yang-Mills condition (Lemma 23 and Corollary 24), in a somewhat similar vein as the classical identification of selfdual and HYM connections on compact Kähler surfaces [DK90, §2]. To conclude with an example, in Section 4.5 we focus on the simplest case of the Fermat quintic CY link, in which E is a pullback from the projective quintic CY^3 , derive the explicit local equations of G_2 -instantons in that setting:

Theorem 4. *Suppose $\pi : K \rightarrow V$ is the 7-dimensional quintic CY link, and let $\mathcal{E} := \pi^* \mathcal{E}_0 \rightarrow K$ be the pullback from a Hermitian vector bundle $\mathcal{E}_0 \rightarrow V$; then*

- (i) *if a connection $\mathbf{A} = A + \sigma\theta$ on E is a G_2 -instanton, then A defines locally a family $\{A_t\}_{t \in S^1}$ of Hermitian Yang-Mills connections on \mathcal{E}_0 , satisfying*

$$\left(\frac{\partial A_t}{\partial t} - d_{A_t} \sigma \right) \wedge \theta = 0.$$

- (ii) *if \mathcal{E} is indecomposable, there is a one-to-one correspondence between S^1 -invariant G_2 -instantons on \mathcal{E} and Hermitian Yang-Mills connections on \mathcal{E}_0 .*

In particular, Theorem 4 implies that S^1 -invariant G_2 -instantons on \mathcal{E} are ‘counted’ by the Donaldson-Thomas invariants of \mathcal{E}_0 , and this count should remain constant at least for any S^1 -invariant deformations of the G_2 -structure (1). Finally, we underscore that the homogeneous case is offered as proof of principle, since our narrative seems to readily extend to crepant resolutions of weighted projective Calabi-Yau 3-orbifolds.

Readers interested in a more detailed account of instanton theory on G_2 -manifolds are kindly referred to the introductory sections of [SE15, SEW15] and citations therein.

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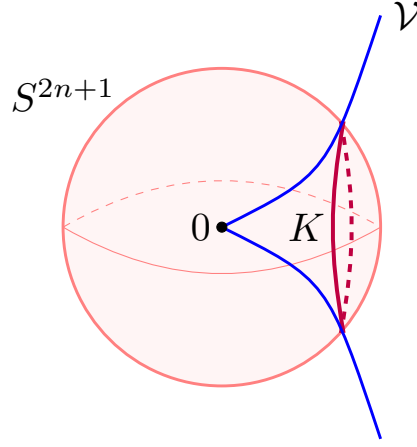
2. GEOMETRIC STRUCTURES ON LINKS

We address the possibilities of G_2 -geometry on Calabi-Yau links, starting from the motivational fact that a 7-manifold admits a G_2 -structure if and only if it is orientable and spin [Gra69], as is the case of links weighted homogeneous hypersurface singularities in \mathbb{C}^5 [BG08, Theorem 9.3.2]. Such links have a very rich tautological geometry, including a null Sasakian structure with a compatible non-degenerate 3-form which is ‘transversely’ holomorphic, fitting in the category of contact Calabi-Yau manifolds proposed by Tomassini and Vezzoni [TV08]. In this section we compile relevant definitions and known properties of weighted homogeneous links, and derive some straightforward consequences.

2.1. Hypersurface links of isolated singularities. We begin by reviewing more carefully Milnor’s fibration theorem, following the original reference [Mil69, §5-7]. We denote by \overline{B}_ε the closed ball of radius ε centered at the origin of \mathbb{C}^{n+1} , by $S_\varepsilon^{2n+1} = \partial B_\varepsilon(0)$ the boundary of this ball, and B_ε for the corresponding open ball. Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a complex analytic map with $f(0) = 0$ and denote $\mathcal{V} := f^{-1}(0)$ and $K_f := \mathcal{V} \cap S_\varepsilon^{2n+1}$ (Figure 1).

Theorem 5. *Let $\varepsilon > 0$ be sufficiently small; then the map*

$$\phi : S_\varepsilon^{2n+1} - K \rightarrow S^1, \quad \phi = \frac{f(x)}{|f(x)|},$$

FIGURE 1. Link K of a hypersurface \mathcal{V} in \mathbb{C}^{n+1}

is a locally trivial fibration, each fibre $F = \phi^{-1}(a)$ is smooth parallelisable and has the homotopy type of a finite CW-complex of dimension n . Furthermore, if f has an isolated singularity at 0, then each fibre F has the homotopy type of a bouquet $S^n \vee \dots \vee S^n$ of spheres, and it is homotopy-equivalent to its closure \overline{F} which is a compact manifold with boundary, with common boundary $\partial \overline{F} = K$. Likewise, K_f is a smooth $(n-2)$ -connected real manifold of dimension $2n-1$.

The number μ of spheres S^n in the bouquet described in Theorem 5 is called the *Milnor number* and it is an extremely important topological invariant of the link.

Theorem 6. *The Milnor number μ has the following interpretations:*

- (i) μ is the complex dimension of the vector space obtained by taking the quotient of the local ring $\mathcal{O}_0(\mathbb{C}^{n+1})$ of holomorphic functions at $0 \in \mathbb{C}^{n+1}$ by the Jacobian ideal $J_f = (\partial f / \partial z_0, \dots, \partial f / \partial z_n)$ of f :

$$\mu = \dim_{\mathbb{C}} \frac{\mathcal{O}_0(\mathbb{C}^{n+1})}{J_f},$$

- (ii) μ is the rank of the free Abelian middle homology group $H_n(F)$,
- (iii) μ is determined by the Euler characteristic of F :

$$\chi(F) = 1 + (-1)^n \mu.$$

The following result [Mil69, Theorem 5.11] gives a useful alternative description of the Milnor fibre:

Theorem 7. *If a complex number $c \neq 0$ is sufficiently close to zero, then the complex hypersurface $f^{-1}(c)$ intersects the open ball B_ε along a smooth manifold which is diffeomorphic to the fibre F .*

Now we focus on the particular case in which f is a weighted homogeneous polynomial with an isolated singularity at $0 \in \mathbb{C}^{n+1}$. This case is special because $\mathcal{V} := f^{-1}(0)$ intersects transversally every sphere S_r^{2n+1} around the origin. Recall the definition of a weighted homogeneous polynomial.

Definition 2. A polynomial $f(z_0, \dots, z_n)$ is called a *weighted homogeneous polynomial* of degree d and weights (w_0, \dots, w_n) if for any $\lambda \in \mathbb{C}^*$

$$(2) \quad f(\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \dots, z_n).$$

NB.: a homogeneous polynomial of degree d is weighted homogeneous of weights $(1, \dots, 1)$.

Proposition 3 ([MO70], Theorem 1). *Let $f(z_0, \dots, z_n)$ be a weighted homogeneous polynomial of degree d and weights (w_0, \dots, w_n) having an isolated singularity at the origin. Then the cohomology $H_n(F, \mathbb{Z})$ is free Abelian of rank $\mu = (\frac{d}{w_0} - 1) \dots (\frac{d}{w_n} - 1)$.*

The Milnor fibration associated to a weighted homogeneous polynomial can appear under a different dressing, as the following lemma shows ([Mil69, Lemma 9.4]; see also [Dim92, Chapter 3, exercises 1.11 and 1.13]).

Lemma 4. *Let $f(z_0, \dots, z_n)$ be a weighted homogeneous polynomial. Then the mapping*

$$f : \mathbb{C}^{n+1} - \mathcal{V} \rightarrow \mathbb{C}^*$$

given by restriction of f is a locally trivial fibration. Denote by ψ the restriction of the above fibration over the unit circle S^1 , then ψ is fibre-diffeomorphic to the Milnor fibration ϕ of Theorem 5 associated to f . In particular the Milnor fibre is diffeomorphic to the non-singular affine hypersurface $\tilde{\mathcal{V}} := \{z \in \mathbb{C}^{n+1} | f(z) = 1\}$.

Weighted homogeneous polynomials give rise in a natural way to links, fibering by circles over weighted projective hypersurfaces:

Definition 5. Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a w -weighted homogeneous polynomial with an isolated critical point at 0, so that each sphere $S^{2n+1} = \partial B_\varepsilon(0)$ intersects $\mathcal{V} := f^{-1}(0) \subset \mathbb{C}^{n+1}$ transversely. Then $K_f := \mathcal{V} \cap S^{2n+1}$ is called a *weighted link of degree $\deg f$ and weight w* .

Given a weight vector $w = (w_0, \dots, w_n)$, denote by $\mathbb{C}^*(w)$ the weighted \mathbb{C}^* -action on \mathbb{C}^{n+1} given by

$$(3) \quad (z_0, \dots, z_n) \rightarrow (\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n).$$

We have the commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & S^9 \\ \downarrow & & \downarrow \\ V & \longrightarrow & \mathbb{P}^4(w), \end{array}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are principal S^1 -orbibundles and orbifold Riemannian submersions. As a complex orbifold, the hypersurface $V \subset \mathbb{P}^4(w)$ can be represented as the quotient $(\mathcal{V} - 0) / \mathbb{C}^*(w)$ where $\mathcal{V} = f^{-1}(0)$.

2.2. G_2 -geometry. We now address the context of Theorem 1, concerning the natural cocalibrated G_2 -structure (1). This section serves the double purpose of recalling notions of G_2 -geometry and setting the scene for the gauge theoretical investigation in Section 4.

Let Y be an oriented smooth 7-manifold. A G_2 -structure is a smooth tensor $\varphi \in \Omega^3(Y)$ identified, at every $p \in Y$, by some frame $f_p : T_p Y \rightarrow \mathbb{R}^7$, with the model (sign conventions of [Sal89])

$$(4) \quad \varphi_0 = e^{567} + \omega_1 \wedge e^5 + \omega_2 \wedge e^6 + \omega_3 \wedge e^7$$

in the sense that $\varphi_p = f_p^*(\varphi_0)$, where

$$\omega_1 = e^{12} - e^{34}, \quad \omega_2 = e^{13} - e^{42}, \quad \text{and} \quad \omega_3 = e^{14} - e^{23}$$

are the canonical generators of selfdual 2-forms in $\Lambda_+^2(\mathbb{R}^4)^*$. The pointwise inner-product

$$\langle u, v \rangle e^{1 \dots 7} := \frac{1}{6} (u \lrcorner \varphi_0) \wedge (v \lrcorner \varphi_0) \wedge \varphi_0$$

determines a Riemannian metric g_φ on Y , under which $*_\varphi \varphi$ is given pointwise by

$$(5) \quad *_\varphi \varphi_0 = e^{1234} - \omega_1 \wedge e^{67} - \omega_2 \wedge e^{75} - \omega_3 \wedge e^{56}.$$

In the language of calibrated geometry [HL82], a 7-manifold with G_2 -structure (Y, φ) is said to be *calibrated* if $d\varphi = 0$ and *cocalibrated* if $d*\varphi = 0$; moreover it is common to omit Y and refer simply to φ in those terms. Cocalibrated G_2 -structures appear in the Fernández-Gray classification [FG82] of G_2 -structures by their intrinsic torsion. A G_2 -structure φ is both calibrated and cocalibrated if and only if $\nabla^{g_\varphi}\varphi = 0$, in which case $\text{Hol}(g_\varphi) \subseteq G_2$ and it is said to be *torsion-free* [Sal89, Lemma 11.5].

Let us consider the following familiar example found, for example, in [Joy00, Proposition 11.1.2]:

Example 6. Let (Z, ω, ε) be a Calabi-Yau 3-fold. Then the product manifold $Z \times S^1$ has a natural torsion-free G_2 -structure defined by:

$$\varphi := dt \wedge \omega + \text{Im } \varepsilon,$$

where t is the variable in S^1 and tensors are denoted identically to their pullbacks under projection onto the Z factor. The Hodge dual of φ is

$$\psi := *\varphi = \frac{1}{2}\omega \wedge \omega + dt \wedge \text{Re } \varepsilon$$

and the induced metric $g_\varphi = g_Z + dt \otimes dt$ is the Riemannian product metric on $Z \times S^1$, with holonomy $\text{Hol}(g_\varphi) = \text{SU}(3)$ properly contained in G_2 .

In the case of a quintic link, we only deviate from the product model of Example 6 in the sense that K_f is necessarily *nontrivial* as a circle bundle over the CY^3 base V , since $\pi_1(K) = \{1\}$ by Theorem 5, so it is fair to ask whether K_f also inherits a ‘globally twisted’ G_2 -structure from the Calabi-Yau structure of V .

Remark 6. If a Lie group G induces a G -structure on a manifold M , then every bundle of tensors splits into summands corresponding to irreducible representations of G . The link K_f carries a G_2 -structure so, in particular, 2-forms split as

$$\Omega^2(K) = \Omega_7^2(K) \oplus \Omega_{14}^2(K),$$

where $\Omega_7^2(K)$ and $\Omega_{14}^2(K)$ are vector subbundles of $\Omega^2(K)$ with fibres isomorphic to the irreducible 7 and 14 representations of G_2 , respectively. It is a well-known fact about manifolds with a G_2 -structure [Bry87, SE15] that $(\Omega^2)_{\frac{7}{14}}$ is respectively the $\frac{-2}{+1}$ -eigenspace of the G_2 -equivariant linear map

$$\begin{aligned} T_\varphi : \Omega^2 &\rightarrow \Omega^2 \\ \eta &\mapsto T_\varphi \eta := *(\eta \wedge \varphi). \end{aligned}$$

2.3. Links as Sasakian 7-manifolds. A *contact manifold* (M, θ) is given by a smooth $(2n+1)$ -manifold M and a *contact structure* $\theta \in \Omega^1(M)$ such that $\theta \wedge (d\theta)^n \neq 0$, everywhere on M . On a contact manifold there exists a unique *Reeb vector field* $\xi \in \Gamma(TM)$, such that $\xi \lrcorner \theta = 1$ and $\xi \lrcorner d\theta = 0$. The Reeb vector field is nowhere-vanishing, so it uniquely determines a 1-dimensional foliation N_ξ called the *characteristic foliation*. It is customary to think of contact manifolds as odd-dimensional analogues of symplectic manifolds, with the 2-form $d\theta$ being ‘transversely symplectic’ with respect to the characteristic foliation. From that perspective, Sasakian geometry encodes the notion of ‘transversely Kähler structure’:

Definition 7. A *Sasakian structure* on M is a quadruple (M, θ, g, Φ) such that (M, g) is a Riemannian manifold, (M, θ) is a contact manifold with Reeb vector field ξ , Φ is a global section of $\text{End}(TM)$, and the following relations hold:

$$\begin{aligned} g(\xi, \xi) &= 1, \quad \Phi \circ \Phi = -\text{Id}_{TM} + \theta \otimes \xi, \quad g(\Phi X, \Phi Y) = g(X, Y) - \theta(X)\theta(Y), \\ \nabla_X^g \xi &= -\Phi X, \quad (\nabla_X^g \Phi)(Y) = g(X, Y)\xi - \theta(Y)X, \end{aligned}$$

where X, Y are vector fields on M and ∇^g is the Levi-Civita connection corresponding to g . If (M, θ, g, Φ) satisfies these conditions we say M is a *Sasakian manifold*.

If the orbits of ξ are all closed, hence circles, then ξ integrates to an isometric $U(1)$ action on M , in particular this action is locally free. If the action is in fact free then the Sasakian structure is said to be *regular*, otherwise, it is said to be *quasi-regular*. The leaf space $\mathcal{Z} := M/N_\xi = M/U(1)$ has the structure of a manifold or orbifold, in the regular or quasi-regular case respectively.

The sphere S^{2n+1} has a natural contact structure

$$\theta_c = -\frac{i}{2} \sum_{j=0}^n (z_j d\bar{z}_j - \bar{z}_j dz_j)$$

given by the Hopf fibration $\mathcal{H} : S^{2n+1} \xrightarrow{\pi} \mathbb{P}^n$; in real coordinates $z_j = x_j + iy_j$ the contact form is $\theta_c = \sum_{i=0}^n y_i dx_i - x_i dy_i$. In fact it is well known [SH62] that S^{2n+1} carries a regular Sasakian structure $(S^{2n+1}, \theta_c, g_c, \Phi_c)$ with Reeb vector field

$$\xi_c = \sum_{i=0}^n \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right) = -i \sum_{j=0}^n \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right),$$

the metric g_c is the flat metric on \mathbb{R}^{2n+1} restricted to S^{2n+1} , and

$$\Phi_c = \sum_{i,j} \{ [(x_i x_j - \delta_{ij}) \partial_{x_i} + (x_j y_i) \partial_{y_i}] \otimes dy_j - [(y_i y_j - \delta_{ij}) \partial_{y_i} + x_i y_j \partial_{x_i}] \otimes dx_j \}.$$

The links of isolated hypersurface singularities admit Sasakian structures in a natural way.

Proposition 8 (Proposition 9.2.2 [BG08]). *Let K_f be the link of a hypersurface singularity. Then the Sasakian structure $\mathcal{S}_c := (\theta_c, g_c, \Phi_c)$ on S^{2n+1} defined above induces by restriction a Sasakian structure, also denoted by \mathcal{S}_c , on the link K_f .*

2.4. Contact Calabi-Yau structures on links. Contact Calabi-Yau manifolds were introduced by Tomassini and Vezzoni in [TV08] and thoroughly studied by Habib and Vezzoni in [HV15], as a development of Reinhart's general theory of Riemannian foliations [Rei59]. This concept describes Sasakian manifolds endowed with a closed basic complex volume form, which is 'transversally holomorphic' in a certain sense (see Definition 10). Most importantly for us, it allows for a vast generalisation of the G_2 -geometry on homogeneous links discussed in Section 2.2.

Let (M, θ) be a contact manifold with contact 1-form θ and denote $B := \ker \theta$ its contact distribution of rank $2n$, i.e., $TM = B \oplus N_\xi$. Let X denote an arbitrary vector field tangent to the characteristic foliation N_ξ . A differential form $\beta \in \Omega^k(M)$ is said to be *transversal* if $X \lrcorner \beta = 0$ and $\mathcal{L}_X \beta = 0$ for every such X .

If (M, θ, Φ) is a Sasakian manifold, and $x \in M$, it follows from Definition 7 that $(\Phi|_{B_x})^2 = -\text{Id}_{B_x}$. Then we can decompose the complexification $B_x \otimes_{\mathbb{R}} \mathbb{C}$ into the eigenspaces of the complexified automorphism $\Phi|_{B_x} \otimes_{\mathbb{R}} \mathbb{C}$:

$$B_x \otimes_{\mathbb{R}} \mathbb{C} = B_x^{1,0} \oplus B_x^{0,1},$$

where $B_x^{1,0}$ and $B_x^{0,1}$ correspond to the eigenvalues $\mathbf{i} := \sqrt{-1}$ and $-\mathbf{i}$ respectively. This induces a splitting of the exterior differential algebra over $B_{\mathbb{C}} := B \otimes_{\mathbb{R}} \mathbb{C}$:

$$(7) \quad \Omega^k(B_{\mathbb{C}}) = \bigoplus_{p+q=k} \Omega^{p,q}(M),$$

where $\Omega^{p,q}(M) := \Gamma(\Lambda^p(B^{1,0})^* \otimes (\Lambda^q B^{0,1})^*)$ and $p, q \geq 0$. Then we have an obvious decomposition of exterior forms on M given by

$$\Omega^j(M) = \bigoplus_{p+q=j} \Omega^{p,q}(M) \oplus \bigoplus_{p+q=j-1} \Omega^{p,q}(M) \wedge \theta.$$

If $\beta \in \Omega^k(M)$ is a transversal differential form, we will say that β is of type (p, q) if it belongs to $\Omega^{p,q}(M)$. The following lemma [BS10, Corollary 3.1] will be crucial for our applications in gauge theory, so we sketch for convenience:

Lemma 9. *Let (M, θ, Φ) be a Sasakian manifold. Then $d\theta \in \Omega^{1,1}(M)$.*

Proof. That $d\theta$ is transversal is clear from Definition 7. It is easy to prove that $d\theta(X, Y) = -g(\Phi X, Y)$ for all $X, Y \in B$, then $d\theta$ is of type $(1, 1)$. \square

Definition 10. A contact Calabi-Yau manifold (cCY) is a quadruple $(M, \theta, \Phi, \varepsilon)$ such that:

- (M, θ, Φ) is a $2n + 1$ -dimensional Sasakian manifold;
- ε is a nowhere vanishing transversal form on $B = \ker(\theta)$ of type $(n, 0)$:

$$\varepsilon \wedge \bar{\varepsilon} = c_n \omega^n, \quad d\varepsilon = 0,$$

where $c_n = (-1)^{n(n+1)/2}(\mathbf{i})^n$ and $\omega := d\theta$. We denote accordingly

$$\operatorname{Re} \varepsilon := \frac{\varepsilon + \bar{\varepsilon}}{2} \quad \text{and} \quad \operatorname{Im} \varepsilon := \frac{\varepsilon - \bar{\varepsilon}}{2\mathbf{i}}.$$

Our interest in cCY structures for G_2 -geometry derives from the following fundamental result:

Proposition 11 ([HV15], subsection 6.2.1). *Let $(M, \theta, \Phi, \varepsilon)$ be 7-dimensional contact Calabi-Yau manifold. Then M carries a transversal co-closed G_2 -structure defined by*

$$(8) \quad \varphi := \theta \wedge \omega + \operatorname{Im} \varepsilon$$

with torsion $d\varphi = \omega \wedge \omega$ (cf. Definition 10) and corresponding dual 4-form

$$\psi = *\varphi = \frac{1}{2}\omega \wedge \omega + \theta \wedge \operatorname{Re} \varepsilon$$

The existence of cCY structures on links is equivalent to a simple numerical criterion on the weighted homogeneous data, which we adopt as a definition:

Definition 12. A weighted link K_f (cf. Definition 5) of degree d and weight $w = (w_0, \dots, w_n)$ is said to be a Calabi-Yau (CY) link if

$$d = \sum_{i=0}^n w_i.$$

The condition $d - \sum_{i=0}^n w_i = 0$ means precisely that the Sasakian structure (K, θ_c, Φ_c) on K_f induced from the canonical Sasakian structure of the sphere S^{2n+1} is null Sasakian, i.e., the (basic) first Chern class of (K, θ_c, Φ_c) vanishes. Recall also this vanishing is exactly the requirement for the weighted projective V to be a Calabi-Yau orbifold [CLS90], thus CY links are nontrivial circle fibrations over Calabi-Yau 3-orbifolds. Furthermore, the Reeb vector field the unit tangent to the $S^1(w)$ -action and the 3-form ε is transversal, so the G_2 -structure (8) is S^1 -invariant. In the terms of Definition 12, Habib and Vezzoni's existence result can be restated as:

Proposition 13 ([HV15], Proposition 6.7). *Every Calabi-Yau link admits a S^1 -invariant contact Calabi-Yau structure.*

The proof of Proposition 13 relies on a Sasakian version of the El Kacimi theorem to prove that any null Sasakian structure on a compact simply-connected manifold can be deformed into a contact Calabi-Yau one. Combining the previous two propositions:

Corollary 14 ([HV15], Corollary 6.8). *Every Calabi-Yau link has a cocalibrated S^1 -invariant G_2 -structure of the form (8).*

3. THE ν INVARIANT OF CALABI-YAU LINKS

For an arbitrary closed 7-manifold with G_2 -structure (Y^7, φ) , Crowley and Nordström define a pair of homotopy invariants $(\nu(\varphi), \xi(\varphi))$ which completely classifies the data, up to diffeomorphism and homotopy, if Y is 2-connected [CN15, Theorem 1.17]. Subsequently this has been refined as an analytic invariant of manifolds with G_2 -metrics [CGN15], and similar ideas also intervene in the authors' topological classification of spin 2-connected 7-manifolds [CN14].

We will be interested in the first invariant $\nu(\varphi)$, which is a \mathbb{Z}_{48} -valued combination of topological data from a compact coboundary 8-manifold with a $\text{Spin}(7)$ -structure (W^8, Ψ) filling (Y, φ) , in the sense that $Y = \partial W$ and $\Psi|_Y = \varphi$:

$$(9) \quad \nu(\varphi) := \chi(W) - 3\sigma(W) \pmod{48}$$

(χ and σ denote the real Euler characteristic and the signature, respectively.) This quantity is preserved under diffeomorphisms of Y and homotopies of the G_2 -structure φ [CN15, Theorem 1.3]. Moreover, $\nu(\varphi)$ is independent of the particular choice of coboundary W [CN15, Corollary 3.2], thus it is interpreted as an “ \hat{A} -defect” from certain integral characteristic classes of principal $\text{Spin}(8)$ -bundles evaluated on TW and the half-spinor bundles $S^\pm W$.

A central aspect is the fact that such a filling W always exists [CN15, Lemma 3.4 (ii)]. The argument relies on the fact that the bordism group Ω_7^{spin} is trivial, hence there always exists *some* (connected) coboundary (W, Ψ) inducing a reference G_2 -structure on Y , but it is totally non-constructive. For example, the authors must resort to an elaborate construction of an explicit coboundary W to calculate $\nu = 24$ [Theorem 1.7] for the important class of manifolds with holonomy G_2 obtained as *twisted connected sums* [CHNP15]. This allows one to distinguish, for instance, whether a given G_2 -structure is not a gluing of asymptotically cylindrical Calabi-Yau 3-folds [CHNP13].

3.1. Construction of a spin coboundary. In order to calculate the ν invariant for our G_2 -structure (8) on a link K_f , we must therefore find an ad hoc compact $\text{Spin}(7)$ -coboundary (W, Ψ) such that:

$$K = \partial W \quad \text{and} \quad \Psi|_K = \varphi.$$

Let K_f be the weighted link (cf. Definition 5) of degree d and weight $w = (w_0, \dots, w_4)$. The ambient 4-form

$$\Psi := \frac{1}{f} \sum_{i=0}^4 z_i dz_0 \wedge \dots \wedge \hat{z}_i \wedge dz_4 \in \Lambda^{4,0}(\mathbb{C}^5)$$

is $S^1 \subset \mathbb{C}^*$ -invariant under the action (3) if, and only if, $d - \sum_{i=0}^4 w_i = 0$, i.e., exactly when the link K_f is Calabi-Yau (Definition 12). Let f_ε be a smoothing of f , e.g. $f_\varepsilon := f - \varepsilon$ and

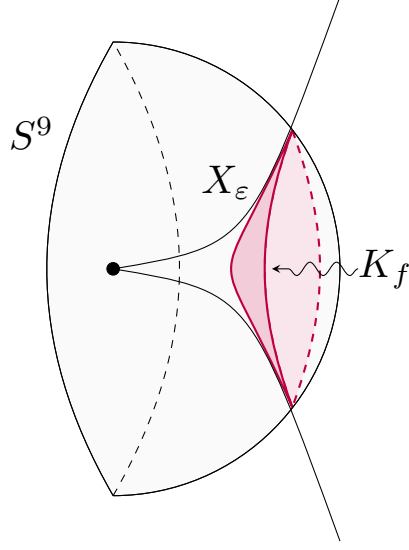
$$X_\varepsilon := f_\varepsilon^{-1}(0) \cap \overline{B}^{10} \subset \mathbb{C}^5$$

the 8-manifold inside the (compact component of the complement of the) sphere S^9 with boundary $K_f = \partial X_\varepsilon$ (Figure 2). The restriction $\Psi|_{X_\varepsilon}$ induces an $SU(4)$ -structure, hence a $\text{Spin}(7)$ -structure on X_ε , which is S^1 -invariant by construction.

Restricting to the boundary we get an S^1 -invariant G_2 -structure on K_f , which corresponds exactly to an $SU(3)$ -structure φ' on the CY^3 quintic V . Now, all $SU(3)$ -structures on a 6-manifold are homotopic, as sections of a bundle of rank 8, so φ' is homotopic to our φ and we can take $W = X_\varepsilon$:

$$(10) \quad \nu(\varphi) = \nu(\varphi') = \chi(X_\varepsilon) - 3\sigma(X_\varepsilon) \pmod{48}.$$

Now all we need is to calculate the topology of the smoothing of an affine hypersurface.

FIGURE 2. Smoothing X_ε of \mathcal{V} inside S^9 , with boundary K_f .

Proposition 15. *Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be weighted homogeneous polynomial with typical Milnor fibre F (cf. Theorem 5), and consider the model affine variety*

$$\tilde{\mathcal{V}} := \{z \in \mathbb{C}^{n+1} : f(z) = 1\}.$$

Given $\varepsilon > 0$ sufficiently small, the smoothing $f_\varepsilon := f - \varepsilon$, defines in \mathbb{C}^{n+1} a compact manifold with boundary $X_\varepsilon := f_\varepsilon^{-1}(0) \cap \overline{B}^{2n+2}$, and the following are diffeomorphic:

$$(11) \quad X_\varepsilon \simeq \overline{F} \quad X_\varepsilon \setminus \partial X_\varepsilon \simeq F \simeq \tilde{\mathcal{V}}.$$

Proof. Taking $\varepsilon = c = 1$ in Theorem 7, we identify diffeomorphically the smoothing X_ε with the closure of the Milnor fibre \overline{F} . Then the second identification is immediate from Lemma 4. \square

3.2. Explicit formula for ν on Calabi-Yau links. In view of Proposition 15, we will obtain the ν invariant from the following:

$$(12) \quad \chi(X_\varepsilon) = \chi(\overline{F}) \quad \text{and} \quad \sigma(X_\varepsilon) = \sigma(\tilde{\mathcal{V}}).$$

We begin with Steenbrink's method [Ste77] for the signature of $\tilde{\mathcal{V}}$. Let $\{z^\alpha : \alpha = (\alpha_0, \dots, \alpha_n) \in I \subset \mathbb{N}^{n+1}\}$ be a set of monomials in $\mathbb{C}[z_0, \dots, z_n]$ representing a basis over \mathbb{C} for $\frac{\mathbb{C}[[z_0, \dots, z_n]]}{(\partial f / \partial z_0, \dots, \partial f / \partial z_n)}$ (cf. (i) of Theorem 6). For each $\alpha \in I$ define

$$(13) \quad l(\alpha) := \sum_{i=0}^n (\alpha_i + 1) \frac{w_i}{d}.$$

Assume that n is even (in our case, indeed $n = 4$), and denote by (μ_-, μ_0, μ_+) the signature of the intersection form on $H^n(\tilde{\mathcal{V}}, \mathbb{R})$ i.e., μ_- , μ_0 and μ_+ denote the numbers of negative, zero and positive entries, respectively, on the diagonal of the intersection matrix. Then

$$\sigma(\tilde{\mathcal{V}}) = \mu_+ - \mu_-.$$

Note that the sum $\mu_+ + \mu_- + \mu_0$ equals the Milnor number μ by (ii) of Theorem 6. On the other hand, by (iii) of Theorem 6, the Euler characteristic of the Milnor fiber is determined

by the Milnor number, which is given by Proposition 3 for weighted homogenous links. By Theorem 5, F is homotopy-equivalent to \overline{F} , so for $n = 4$:

$$\chi(\overline{F}) = \chi(F) = 1 + \left(\frac{d}{w_0} - 1\right) \dots \left(\frac{d}{w_4} - 1\right).$$

Finally, replacing (12) in (10), we establish the formula of Theorem 2:

$$(14) \quad \nu(\varphi) = \left(\frac{d}{w_0} - 1\right) \dots \left(\frac{d}{w_4} - 1\right) - 3(\mu_+ - \mu_-) + 1.$$

Steenbrink proved [Ste77, Theorem 2] that the signature (μ_-, μ_0, μ_+) can be computed as follows:

$$\begin{aligned} \mu_+ &= |\{\beta \in I : l(\beta) \notin \mathbb{Z}, \lfloor l(\beta) \rfloor \in 2\mathbb{Z}\}|, \\ \mu_- &= |\{\beta \in I : l(\beta) \notin \mathbb{Z}, \lfloor l(\beta) \rfloor \notin 2\mathbb{Z}\}|, \\ \mu_0 &= |\{\beta \in I : l(\beta) \in \mathbb{Z}\}|, \end{aligned}$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{Q}$, hence the above process can be easily implemented. We offer the code for a working algorithm in a combination of SINGULAR and MATHEMATICA, but surely readers will be able to formulate leaner alternatives. We display in Table 1 the invariants given by (14) for some examples from Candelas' list of weighted Calabi-Yau threefolds.

TABLE 1. The ν invariant for certain Calabi-Yau links

degree	weights	polynomial	ν
75	(10,12,13,15,25)	$z_0^5 z_4 + z_1^5 z_3 + z_2^5 z_0 + z_3^5 + z_4^3$	1
135	(1,18,32,39,45)	$z_0^{135} + z_1^5 z_4 + z_2^3 z_3 + z_3^3 z_1 + z_4^3$	3
36	(18, 12, 4, 1, 1)	$z_0^2 + z_1^3 + z_2^9 + z_3^{36} + z_4^{36}$	5
81	(3,7,18,26,27)	$z_0^{27} + z_1^9 z_2 + z_2^3 z_4 + z_3^3 z_0 + z_4$	7
45	(3,5,8,14,15)	$z_0^{15} + z_1^9 + z_2^5 z_1 + z_3^3 z_0 + z_4^3$	9
45	(4,7,9,10,15)	$z_0^9 z_2 + z_2^5 + z_1^5 z_3 + z_3^3 z_4 + z_4^3$	11
75	(5,8,12,15,35)	$z_0^{15} + z_1^5 z_4 + z_2^5 z_3 + z_3^5 + z_4^2 z_0$	13
180	(90, 60, 20, 9, 1)	$z_0^2 + z_1^3 + z_2^9 + z_3^{20} + z_4^{180}$	15
45	(15, 15, 5, 9, 1)	$z_0^3 + z_1^3 + z_2^9 + z_3^5 + z_4^{45}$	17
16	(4,8,2,1,1)	$z_0^2 z_1 + z_1^2 + z_2^4 z_1 + z_3^{16} + z_4^{16} + z_2^8$	19
81	(2,9,19,24,27)	$z_0^{27} z_4 + z_2^3 z_3 + z_3^3 z_1 + z_1^9 + z_4^3$	21
24	(12, 8, 2, 1, 1)	$z_0^2 + z_1^3 + z_2^{12} + z_3^{24} + z_4^{24}$	23
1806	(42, 258, 903, 602, 1)	$z_0^{43} + z_1^7 + z_2^2 + z_3^3 + z_4^{1806}$	25
51	(2,6,9,17,17)	$z_0^{17} z_4 + z_1^7 z_2 + z_2^5 z_1 + z_3^3 + z_4^3$	29
93	(3,8,21,30,31)	$z_0^{31} + z_1^9 z_2 + z_2^3 z_3 + z_3^3 z_0 + z_4^3$	31
63	(3,4,14,21,21)	$z_0^{21} + z_1^{15} z_0 + z_2^3 z_3 + z_3^3 + z_4^3$	33
103	(1,16,23,29,34)	$z_0^{103} + z_1^5 z_2 + z_2^3 z_4 + z_3^3 z_1 + z_4^3 z_0$	37
135	(5,6,14,45,65)	$z_0^{27} + z_4^2 z_0 + z_1^{15} z_3 + z_3^3 + z_2^5 z_4$	39
60	(30, 20, 5, 4, 1)	$z_0^2 + z_1^3 + z_2^{12} + z_3^{15} + z_4^{60}$	41
55	(4,4,11,17,19)	$z_0^{11} z_2 + z_1^9 z_4 + z_2^5 + z_3^3 z_1 + z_4^2 z_3$	43
135	(1,21,30,38,45)	$z_0^{135} + z_1^5 z_2 + z_2^3 z_4 + z_3^3 z_1 + z_4^3$	45
45	(5, 5, 9, 11, 12)	$z_0^9 + z_1^8 z_0 + z_2^5 + z_4^3 z_2 + z_3^3 z_4$	47

Inspection of a few examples suggests a parity constraint for the ν invariant, and this is indeed the case:

Proposition 16. *The Crowley-Nordström ν invariant of a weighted link is odd in \mathbb{Z}_{48} .*

Proof. We know from [CN15, Theorem 1.3] that $\nu(\varphi) \equiv \chi_{\mathbf{Q}}(K) \pmod{2}$, where $\chi_{\mathbf{Q}}(K) := \sum_{i=0}^n b_i(K)$ is the rational semi-characteristic of K_f . On the other hand, $b_1 = b_2 = 0$ because K_f is 2-connected (cf. Theorem 5), and we know from [BG08, Theorem 9.3.2] that the Betti number b_{n-1} is even, if n is even. Therefore b_3 is even when $n = 4$, thus $\chi_{\mathbf{Q}}(K)$ is odd. \square

Together with formula (14), this completes the proof of Theorem 2.

Example 17. Let us calculate the ν invariant for our G_2 -structure (1) on the Fermat quintic

$$f(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5.$$

In this case the Milnor algebra is just $\frac{\mathbb{C}[z_0, \dots, z_4]}{(z_0^4, \dots, z_4^4)}$, the Milnor number is $\mu = 1024$ and we can take, as a basis of the Milnor algebra, all monomials of the form $z_0^{\alpha_0} \dots z_4^{\alpha_4}$, with $0 \leq \alpha_i \leq 3, \forall i$. A simple computation gives

$$(\mu_-, \mu_0, \mu_+) = (240, 204, 580),$$

therefore $\sigma(\tilde{\mathcal{V}}) = 340$. On the other hand, Theorem 6 gives $\chi(X_\varepsilon) = 1 + \mu = 1025$. Hence $\nu(\varphi) = 5$ as claimed in Corollary 1.

4. GAUGE THEORY ON CONTACT CALABI-YAU 7-MANIFOLDS

Let (M, φ) be a closed contact Calabi-Yau manifold and consider a G -bundle $E \rightarrow M$ with G a compact semisimple Lie group, denote by $\mathcal{G} := \Gamma(\text{Aut } E)$ its gauge group with $\mathfrak{g} := \text{Lie}(\mathcal{G})$ the associated adjoint bundle and by $\mathcal{A}(E)$ its space of connections. We address the classical problem of describing the absolute minima of the *Yang-Mills functional*

$$\begin{aligned} \mathcal{S}_{\text{YM}} &: \mathcal{A}(E) \rightarrow \mathbb{R}^+ \\ \mathcal{S}_{\text{YM}}(A) &:= \|F_A\|_{L^2(M)}^2 = \int_M \langle F_A \wedge *F_A \rangle_{\mathfrak{g}} \end{aligned}$$

i.e., solutions of the *Yang-Mills equation*:

$$(15) \quad d_A^* F_A = 0.$$

4.1. Yang-Mills connections, G_2 -instantons and the Chern-Simons action. The paradigmatic PDE for gauge theory in the presence of a G_2 -structure is the G_2 -instanton equation [DT98, Tia00], which can be formulated equivalently in terms of φ or $\psi := *\varphi$:

$$(16) \quad F_A \wedge \psi = 0 \quad \Leftrightarrow \quad *F_A = F_A \wedge \varphi.$$

This is the natural Euler-Lagrange equation for the *Chern-Simons action*, defined relatively to a fixed reference connection $A_0 \in \mathcal{A}(E)$ by

$$\begin{aligned} \mathcal{S}_{\text{CS}} &: \mathcal{A}(E) \simeq A_0 + \Omega^1(\mathfrak{g}) \rightarrow \mathbb{R} \\ \mathcal{S}_{\text{CS}}(A_0 + a) &:= \frac{1}{2} \int_M \text{Tr} \left(d_{A_0} a \wedge a + \frac{2}{3} a \wedge a \wedge a \right) \wedge \psi \end{aligned}$$

with $\mathcal{S}_{\text{CS}}(A_0) = 0$. Supposing the cocalibrated condition $d\psi = 0$, the action is well-defined and G_2 -instantons are manifestly critical points. Its gradient is the *Chern-Simons 1-form* $\rho = d\mathcal{S}_{\text{CS}}$, defined on vector fields $b: \mathcal{A}(E) \rightarrow \Omega^1(\mathfrak{g})$ by

$$(17) \quad \rho(b)_A = \int_M \text{Tr} (F_A \wedge b_A) \wedge \psi.$$

and indeed the solutions of (16) are precisely its zeroes (for a more detailed exposition, see [SE14].)

Now, if the G_2 -structure was closed, then by the Bianchi identity every solution of (16) would automatically solve (15). In other words, G_2 -instantons would manifestly be critical points of the Yang-Mills functional, somewhat in analogy to (anti-)selfdual connections in

dimension 4 [DK90]. This indeed was the starting point of our predecessors in proposing gauge theory on G_2 -manifolds. Since then, such Yang-Mills G_2 -instantons have been constructed on Joyce manifolds [Wal13], Bryant-Salamon manifolds [Cla14], associative fibrations [SE14], asymptotically cylindrical G_2 -manifolds [SE09, SE15] and their twisted connected sums [SEW15, Wal15]. However, the implication (16) \Rightarrow (15) depends on a certain characteristic class in de Rham cohomology that yields Chern-Weil energy bounds, and it fails in general for merely cocalibrated G_2 -structures. This is unfortunate, because many examples of the latter are now known (see e.g. [CS06, AF10, Lot12, Fre13] and references therein). As we will show in Section 4.3, a suitable version of the argument *does* hold for the natural cocalibrated G_2 -structure on contact Calabi-Yau manifolds and in particular for Calabi-Yau links, so in our case G_2 -instantons are exactly absolute minima of the Yang-Mills functional, as well as critical points of the natural Chern-Simons action.

4.2. Sasakian vector bundles. Interpreting Sasakian structures as a setup for ‘transversely Kähler’ geometry, any compatible formulation of gauge theory requires a good notion of transversely compatible structures on vector bundles. We adopt the lexicon proposed by Biswas and Schumacher [BS10, §3.3].

Let $E \rightarrow M$ be a C^∞ complex vector bundle over a smooth manifold and let

$$(18) \quad L \subset TM_{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$$

be an integrable subbundle, i.e., closed under the Lie bracket. A *partial connection along* L is a C^∞ differential operator

$$D_L : E \longrightarrow L^* \otimes E$$

satisfying the Leibniz condition for $f \in C^\infty(M)$ and $s \in \Gamma(E)$:

$$D_L(fs) = fD_L(s) + q_L(df)$$

relative to the dual $q_L : T^*M_{\mathbb{C}} \rightarrow L^*$ of the inclusion (18). Since L is integrable, q_L induces a natural exterior derivative $\hat{d} : L^* \rightarrow \bigwedge^2 L^*$, hence an extension of D_L to E -valued sections of L^*

$$\begin{aligned} D_L : L^* \otimes E &\longrightarrow (\bigwedge^2 L^*) \otimes E \\ D_L(t \otimes s) &:= \hat{d}(t) \otimes s - t \otimes D_L(s). \end{aligned}$$

The *curvature* of D_L is the $C^\infty(M)$ -linear section

$$F_{D_L} := D_L^2 \in \Gamma\left((\bigwedge^2 L^*) \otimes \mathfrak{g}\right)$$

and D_L is said to be *flat* if $F_{D_L} = 0$.

Definition 18. A *Sasakian (vector) bundle* $\mathbf{E} \rightarrow M$ over a Sasakian manifold (M, θ, g, Φ) with Reeb field ξ is a pair (E, D_ξ) given by a C^∞ complex vector bundle E over M and a partial connection $D_\xi : E \rightarrow \theta \otimes E$ along ξ .

To be completely precise, we are applying the previous discussion to the line subbundle $L = \xi_{\mathbb{C}} := N_\xi \otimes_{\mathbb{R}} \mathbb{C} \subset TM_{\mathbb{C}}$ spanned by ξ over \mathbb{C} (cf. Section 2.4); it is clear that any such D_ξ is flat. Moreover, the natural partial connection induced by D_ξ on E^* gives natural definitions of \mathbf{E}^* and $\text{End}(\mathbf{E})$ as Sasakian bundles, and we define a *Hermitian structure* on \mathbf{E} as a smooth Hermitian structure h on E preserved by D_ξ , in the sense that

$$d(h(s_1, s_2))|_{\xi_{\mathbb{C}}} = h(D_\xi(s_1), s_2) + h(s_1, D_\xi(s_2)).$$

Clearly a Hermitian structure on \mathbf{E} induces a Hermitian structure on \mathbf{E}^* and on $\text{End}(\mathbf{E})$. A *unitary connection* on (\mathbf{E}, h) is a connection A on E such that d_A preserves h in the usual sense.

Finally, in the notation of (7), we obtain a natural notion of transversely holomorphic structures over a Sasakian manifold M^{2n+1} relative to the integrable ‘extended anti-holomorphic’ $(n+1)$ -dimensional foliation

$$(19) \quad \tilde{B}^{0,1} := B^{0,1} \oplus \xi_{\mathbb{C}} \subset TM_{\mathbb{C}}.$$

Definition 19. A holomorphic (Sasakian) bundle $\mathcal{E} \rightarrow M$ over a Sasakian manifold M with Reeb field ξ is a pair $(\mathbf{E}, \bar{\partial})$ given by a Sasakian bundle $\mathbf{E} = (E, D_{\xi})$ (cf. Definition 18) and a flat partial connection $\bar{\partial} = D_{\tilde{B}^{0,1}}$ such that $\bar{\partial}|_{\xi_{\mathbb{C}}} = D_{\xi}$.

An integrable connection on $\mathcal{E} = (\mathbf{E}, \bar{\partial})$ is connection A on E such that its induced partial connection along $\tilde{B}^{0,1}$ given by $D_{\tilde{B}^{0,1}} := d_A|_{\tilde{B}^{0,1}}$ coincides with $\bar{\partial}$. We denote by $\mathcal{A}(\mathcal{E})$ the subset of integrable connections inside $\mathcal{A}(E)$. We are now in position to extend the well-known concept of a Chern connection, mutually compatible with the holomorphic structure and the Hermitian metric [DK90, Proposition 2.1.56]:

Proposition 20 ([BS10], p.552). *Let (\mathcal{E}, h) be a holomorphic Sasakian bundle with Hermitian structure; then there exists a unique unitary and integrable Chern connection A_h on \mathcal{E} , and*

$$F_{A_h} \in \Omega^{1,1}(\mathfrak{g}).$$

Moreover, the expression

$$(20) \quad \det \left(\text{id}_E + \frac{\mathbf{i}}{2\pi} F_{A_h} \right) =: \sum_{j=0}^n c_j(\mathcal{E}, h)$$

defines closed Chern forms $c_j(\mathcal{E}, h) \in \Omega^{j,j}(M)$.

In a local holomorphic trivialization τ , i.e., such that \mathcal{E} is locally spanned by sections in $\ker \bar{\partial}$, the Chern connection of h is represented by the matrix of $(1,0)$ -forms $A_h^{\tau} = h^{-1} \partial h$ and its curvature has the form $F_{A_h}^{\tau} = \bar{\partial}(h^{-1} \partial h)$. Moreover, it is clear from (19) and Definition 19 that any other Hermitian structure h' induces a Chern connection on \mathcal{E} satisfying $A_{h'} - A_h \in \Omega^{1,0}(\mathfrak{g})$.

4.3. Characteristic classes and topological energy bounds. We will show that Calabi-Yau links admit a naturally defined secondary characteristic class representing topological charge, which is a peculiar feature among G_2 -structures with torsion. From the perspective of gauge theory, this means that critical points of the Chern-Simons functional indeed saturate the Yang-Mills energy, just like in classical 4-dimensional theory or more familiar torsion-free higher dimensional models.

Definition 21. Let $E \rightarrow M$ be a Sasakian bundle (cf. Definition 19) over a 7-dimensional closed contact Calabi-Yau manifold (cf. Definition 10) with G_2 -structure (8) given by Proposition 11. We define the charge of a connection $A \in \mathcal{A}(E)$ by

$$(21) \quad \kappa(A) := \int_M \text{Tr} F_A^2 \wedge \varphi$$

Lemma 22. *In the context of Proposition 20 and Definition 21, fix a Hermitian metric h on \mathcal{E} with Chern connection $A = A_h \in \mathcal{A}(\mathcal{E})$; then any connection on the underlying Sasakian bundle E has the form $A' = A + b$, for some $b \in \Omega^1(\mathfrak{g})$, and the following hold:*

(i) *If $b = \alpha \cdot \theta$ for some $\alpha \in \Omega^0(\mathfrak{g})$, then*

$$\kappa(A') = \kappa(A) + \int_M \text{Tr}(\alpha \cdot F_A) \wedge (d\theta)^2 \wedge \theta$$

In particular, if $\alpha = \lambda \cdot \text{id}_E$ is a homothety, then $\kappa(A') = \kappa(A) + \lambda \cdot c_1(\mathcal{E}, h)$.

(ii) *If $b \in \Omega^{1,0}(\mathfrak{g})$, then $\kappa(A') = \kappa(A)$, hence the charge of any Chern connection is independent of the Hermitian structure and it defines a topological charge $\kappa(\mathcal{E})$.*

Proof. Given any connection $A \in \mathcal{A}(E)$ and variation $b \in \Omega^1(\mathfrak{g})$, we know from standard Chern-Weil theory that

$$\mathrm{Tr} F_{A+b}^2 - \mathrm{Tr} F_A^2 = d(\mathrm{Tr} \eta)$$

for some $\eta \in \Omega^1(\mathfrak{g})$ of the form

$$\eta = \eta(A, b) := F_A \wedge b + \frac{1}{2} d_A b \wedge b + \frac{1}{3} b \wedge b \wedge b.$$

Since by assumption M is a closed manifold, the quantity (21) is defined up to a term given by Stokes' theorem after integration by parts:

$$(22) \quad \int_K \mathrm{Tr} \eta \wedge d\varphi = \int_K \mathrm{Tr} \left(F_A \wedge b + \frac{1}{3} b \wedge b \wedge b \right) \wedge d\varphi.$$

Recall from Lemma 9 and Proposition 11 that $d\varphi = (d\theta)^2 \in \Omega^{2,2}(M)$. Moreover, since A is a Chern connection, Proposition 20 specifies the bi-degree of

$$F_A \wedge d\varphi \in \Omega^{3,3}(\mathfrak{g}).$$

In the situation of (i), the cubic term on the right-hand side of (22) vanishes trivially, since $\theta^2 = 0$, which yields the claim. As for (ii), both terms vanish by excess in bi-degree. \square

Now, following a classical argument, on one hand we have the orthogonal decomposition of the Yang-Mills functional:

$$(23) \quad \mathcal{S}_{\mathrm{YM}}(A) = \|F_A\|^2 = \|F_7\|^2 + \|F_{14}\|^2.$$

On the other hand, applying the G_2 -equivariant eigenspace decomposition from Remark 6 to integrable connections as in (ii) of Lemma 22, a straightforward calculation relates the topological charge to these components:

$$\kappa(\mathcal{E}) = -2 \|F_7\|^2 + \|F_{14}\|^2.$$

Combining with (23), we can isolate the topological charge as a lower bound of the Yang-Mills energy among integrable connections:

$$(24) \quad \mathcal{S}_{\mathrm{YM}}|_{\mathcal{A}(\mathcal{E})}(A) = -\frac{1}{2} \kappa(\mathcal{E}) + \frac{3}{2} \|F_{14}\|^2 = \kappa(\mathcal{E}) + 3 \|F_7\|^2.$$

Hence, if $\mathcal{S}_{\mathrm{YM}}$ attains on $\mathcal{A}(\mathcal{E})$ its absolute topological minimum, this occurs at a connection whose curvature lies either in Ω_7^2 or in Ω_{14}^2 . Moreover, since $\mathcal{S}_{\mathrm{YM}} \geq 0$, the sign of $\kappa(\mathcal{E})$ obstructs the existence of one type or the other, so we fix $\kappa(\mathcal{E}) \geq 0$, compatibly with the existence of our G_2 -instantons (16) with $F_7 = 0$, i.e., such that $\mathcal{S}_{\mathrm{YM}}(A) = \kappa(\mathcal{E})$. We have thus proved Theorem 3.

In summary, among compatible connections, G_2 -instantons (16) over contact Calabi-Yau 7-manifolds are Yang-Mills minima, even though the natural G_2 -structure is not closed.

4.4. G_2 -instantons and the Hermitian Yang-Mills condition. A connection A on a complex vector bundle over a Kähler manifold is *Hermitian Yang-Mills (HYM)* if

$$\hat{F}_A := (F_A, \omega) = 0 \quad \text{and} \quad F_A^{0,2} = 0.$$

This notion extends literally to Sasakian bundles $E \rightarrow M$, taking $\omega = d\theta \in \Omega^{1,1}(M)$ as the transverse Kähler form. Fixing a holomorphic structure on E , it is easy to check that compatible HYM connections are exactly G_2 -instantons:

Lemma 23. *Let \mathcal{E} be a holomorphic bundle over a 7-dimensional closed contact Calabi-Yau manifold M endowed with its natural G_2 -structure (8). Then a Chern connection A on \mathcal{E} is HYM if, and only if, it is a G_2 -instanton.*

Proof. A Chern connection A satisfies $F_A \in \Omega^{1,1}(M)$ (Proposition 20), so taking account of the bidegree of the transverse holomorphic volume form (cf. Definition 10) we have $F_A \wedge \varepsilon = F_A \wedge \bar{\varepsilon} = 0$. Therefore

$$F_A \wedge \operatorname{Im} \varepsilon = \frac{1}{2i} F_A \wedge (\varepsilon - \bar{\varepsilon}) = 0.$$

Now, taking the product with the 4-form we have

$$F_A \wedge \psi = \frac{1}{2} F_A \wedge \omega \wedge \omega = (cst.) \hat{F}_A(*\theta),$$

hence A is a solution of (16) if, and only if, $\hat{F}_A = 0$. \square

This result generalises the well-known fact that HYM connections compatible with a fixed holomorphic structure over a smooth Calabi-Yau 3-fold pull back bijectively to S^1 -invariant G_2 -instantons over the product $CY^3 \times S^1$ [SE15, Proposition 8]. Indeed, it is easy to deduce the corresponding claim for arbitrary circle fibrations:

Corollary 24. *Let X be a Calabi-Yau threefold, let $\pi : Y \rightarrow X$ be a circle fibration endowed with the natural G_2 -structure (1), and let $\mathcal{E} := \pi^* \mathcal{E}_0 \rightarrow Y$ be the pullback from a holomorphic vector bundle $\mathcal{E}_0 \rightarrow X$. Then \mathcal{E} is a holomorphic Sasakian bundle, and a Chern connection A on \mathcal{E}_0 is HYM if, and only if, $\pi^* A$ is a G_2 -instanton on \mathcal{E} .*

Proof. The contact Calabi-Yau structure is trivially given by the global angular form $\theta \in \Omega^1(Y)$ and the pullbacks of the Calabi-Yau data from X , under π , with natural Reeb field determined by $\theta(\xi) = 1$, tangent to the S^1 -action. Then the underlying complex vector bundle $\pi^* \mathcal{E}_0$ is trivial along ξ and we can adopt $D_\xi = d_\xi$ the trivial vertical connection, which is manifestly flat. This defines a Sasakian bundle structure (cf. Definition 18). Moreover, the 6-dimensional distribution $B := \ker \theta \subset TY$ maps under π_* isomorphically to TX , which induces a natural bi-degree decomposition $B = \bigoplus B^{i,j}$. It is immediate to check that the holomorphic structure $\bar{\partial}_0$ on \mathcal{E}_0 pulls back to a holomorphic structure $\bar{\partial} := \pi^* \bar{\partial}_0$ on \mathcal{E} . \square

This gives a correspondence

$$\left\{ \begin{array}{c} S^1\text{-invariant unitary} \\ \text{connections on } \mathcal{E} = \pi^* \mathcal{E}_0 \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{HYM Chern} \\ \text{connections on } \mathcal{E}_0 \end{array} \right\}$$

which proves part (ii) of Theorem 4. Notice that the right-hand side is bijectively parametrised by stable holomorphic structures on the underlying complex vector bundle $\mathcal{E}_0 \rightarrow X$, by the Hitchin-Kobayashi correspondence.

At this point it is sharply relevant to ask whether a Sasakian version of that correspondence may be obtained, via a suitable notion of transverse stability. While Biswas and Schumacher do outline some progress in that direction [BS10, §3.4], we believe the subject still deserves some further elaboration - perhaps as a general theory for transversely Kähler foliated geometries - which shall be the object of future work.

4.5. Example: G_2 -instantons on pullback bundles. Motivated by Corollary 24 in the previous section, let us explore the simplest model case for gauge theory on a 7-dimensional CY link. Let $\pi : Y = K_f \rightarrow V$ be the Fermat quintic link (cf. Example 17), which fibres nontrivially by circles over the smooth 3-fold $V = (f) \subset \mathbb{P}^4$, and consider the holomorphic Sasakian bundle given by pullback $\mathcal{E} := \pi^* \mathcal{E}_0 \rightarrow K_f$ of a holomorphic bundle over V . We would like to describe the explicit local form of the constraint imposed on a connection $\mathbf{A} \in \mathcal{A}(\mathcal{E})$ by the G_2 -instanton equation (16).

Over a trivialising neighbourhood of K_f as a circle fibration, i.e. an open set $U \subset V$ such that $K_f \supset \pi^{-1}(U) \simeq S^1 \times U$, given points $y \in \pi^{-1}(U)$ and $x = \pi(y) \in U$, an arbitrary integrable connection \mathbf{A} on \mathcal{E} can be written as

$$\mathbf{A}(y) \stackrel{\text{loc}}{=} \pi^* A_t(x) + \sigma(x, t) \theta$$

where $\{A_t\}_{t \in S^1}$ is a family of connections on \mathcal{E}_0 and $\sigma \in \Omega^0(K, \mathfrak{g})$, where $\mathfrak{g} := \pi^* \mathfrak{g}_{\mathcal{E}_0}$ is the corresponding adjoint bundle of \mathcal{E} . Let us denote this fact informally by

$$\mathbf{A} = A_t + \sigma\theta.$$

The curvature of \mathbf{A} is the gauge-covariant global 2-form

$$F_{\mathbf{A}} = F_{A_t} + \left(d_{A_t} \sigma - \frac{\partial A_t}{\partial t} \right) \wedge \theta \in \Omega^2(K, \mathfrak{g}).$$

and, replacing that expression in the G_2 -instanton equation (16), one obtains in particular

$$\hat{F}_{A_t}(*\theta) = F_{A_t} \wedge \omega^2 = 0.$$

This is exactly the HYM condition on each A_t . On the other hand, by Proposition 3, if \mathbf{A} is an integrable G_2 -instanton, then it minimises the Yang-Mills functional (23). This implies

$$\left\| \left(d_{A_t} \sigma - \frac{\partial A_t}{\partial t} \right) \wedge \theta \right\|^2 = 0,$$

since otherwise the pullback component A_t alone would violate the minimum topological energy (24):

$$\mathcal{S}_{\text{YM}}(A_t) = \|F_{A_t}\|^2 < \|F_{\mathbf{A}}\|^2 = \mathcal{S}_{\text{YM}}(\mathbf{A}) = \kappa(\mathcal{E}).$$

Moreover, if the family $A_t \equiv A_{t_0}$ is constant, i.e., S^1 -invariant, then $d_{A_{t_0}} \sigma = 0$ implies $\sigma \equiv 0$, since by assumption \mathcal{E} is indecomposable and therefore does not admit nonzero parallel sections, and so \mathbf{A} is indeed a pullback. If the moduli space $\hat{\mathcal{M}}$ of HYM connections on the base V is discrete, then by continuity the family $\{A_t\}$ is contained in a gauge orbit. This concludes the proof of Theorem 4.

Remark 25. Since the moduli space of stable holomorphic bundles on a Fermat quintic Calabi-Yau 3-fold V is known to be discrete, we infer that S^1 -invariant G_2 -instantons should be counted in some sense by the Donaldson-Thomas invariant of V , which is deformation-invariant because $h^{0,2}(V) = 0$ [Tho00, Definition 3.34]. Thus we envisage a ‘conservation of number’ property for S^1 -invariant G_2 -instantons over such Fermat quintic links, to be made precise in upcoming work.

AFTERWORD: ATIYAH’S CONJECTURE AND SINGULAR G_2 -METRICS

Atiyah predicted that the Casson invariant $\lambda(\Sigma)$ of a homology sphere which is the link of a normal complete intersection singularity equals $\frac{1}{8}\sigma(F)$, where F is the Milnor fibre. This was verified for Brieskorn spheres by Fintushel and Stern [FS90], and Neumann and Wahl [NW90] inductively use that fact to confirm the conjecture for weighted homogeneous surface singularities and for links of hypersurfaces of the form $f(x, y) + z^n = 0$, among others. Their theorem suggests a general relation between the Floer homology (or at least the Casson invariant) of a link in \mathbb{C}^3 and the signature of F . Arnold and Floer [Arn95] suggested higher-dimensional analogues, which would require extra structure on the links (e.g. CR or contact structure) and Milnor fibre (e.g. symplectic structure).

In our context, Chern-Simons theory (17) suggests thinking of G_2 -instantons as 7-dimensional analogues of flat connections. Applying the above intuition to the holomorphic Casson invariant of R. Thomas over a CY 3-fold base [Tho00], we wonder whether a version of Atiyah’s conjecture may hold for CY links.

Finally, from the perspective of M-theory, examples of compact G_2 -metrics with prescribed singularities might be within reach, starting from some suitably singular CY link and taking adiabatic limits on the circle fibres near an orbifold singularity. As we have shown, meaningful Yang-Mills theory results may be established on such spaces, even though the G_2 -structure has some torsion.

APPENDIX A. ALGORITHM FOR STEENBRINK'S SIGNATURE THEOREM

As discussed in Section 3.2, Steenbrink's method for the signature of a compactified affine variety depends solely on computing the nonzero signature (μ_+, μ_-) . This requires an explicit basis of the Milnor algebra, which several computational tools provide. We use the following code in SINGULAR [GPS01].

We first compute the numbers (13), for a given polynomial f of degree d and a list of weights W , and arrange them into list L :

```
proc Signature(poly f, list W, int d)
{
  ring A = 0, (a,b,c,d,e), lp;
  list L;
  int s;
  ideal J = jacob(f);
  J = groebner(J);
  ideal K = kbase(J);
  s=size(K);
  for ( int j=1; j <= s; j++ )
  { L[j]=(1+leadexp(K[j])[1])*(W[1]/d)
  + (1+leadexp(K[j])[2])*(W[2]/d)+ (1+leadexp(K[j])[3])*(W[3]/d)
  + (1+leadexp(K[j])[4])*(W[4]/d)+(1+leadexp(K[j])[5])*(W[5]/d);
  }
  return(L);
  write("list.txt", L);
}
```

Then we use MATHEMATICA code to compute the ν invariant from the list L :

```
 $\nu = \text{Mod}[\text{Length}[L]+1-3*(\text{Length}[\text{Select}[\text{Select}[L, \# \setminus \{\text{NotElement}\} \text{Integers \&}], \text{Mod}[\text{IntegerPart}[\#], 2] == 0 \&]] - \text{Length}[\text{Select}[\text{Select}[L, \# \setminus \{\text{NotElement}\} \text{Integers \&}], \text{Mod}[\text{IntegerPart}[\#], 2] == 1 \&]]), 48]$ 
```

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